Applied Mathematics Letters 23 (2010) 121–127

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

An application of fixed point theorem to best approximation in locally convex space

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ARTICLE INFO

Article history:
Received 29 July 2007
Received in revised form 8 May 2009
Accepted 8 June 2009

Keywords:
Best approximation
Demiclosed mapping
Fixed point
Nonexpansive mapping
Locally convex space

ABSTRACT

A common fixed point theorem of Jungck [G. Jungck, On a fixed point theorem of fisher and sessa, Internat. J. Math. Math. Sci., 13 (3) (1990) 497–500] is generalized to locally convex spaces and the new result is applied to extend a result on best approximation.

1. Introduction

During the last four decades several interesting and valuable results were studied extensively in the field of fixed point theorems.

In 1990, Jungck [1] obtained the following theorem for compatible mapping:

Theorem 1.1 (*[1]*). *Let $\mathcal{T}$ and $\mathcal{I}$ be compatible self-maps of a closed convex subset $\mathcal{M}$ of a Banach space $\mathcal{X}$. Suppose $\mathcal{I}$ is linear, continuous, and that $\mathcal{T}(\mathcal{M}) \subseteq \mathcal{I}(\mathcal{M})$. If there exists $a \in (0, 1)$ such that $x, y \in \mathcal{M}$*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq a\|\mathcal{I}x - \mathcal{I}y\| + (1 - a)\max\{\|\mathcal{T}x - \mathcal{I}x\|, \|\mathcal{T}y - \mathcal{I}y\|\}, \tag{1.1}$$

then $\mathcal{T}$ and $\mathcal{I}$ have a unique common fixed point in $\mathcal{M}$.

In this paper, we first derive a common fixed point result in locally convex space which generalizes the result of Jungck [1]. This new result is used to prove another fixed point result for best approximation. By doing so, we in fact, extend and improve the result of Brosowski [2], Meinardus [3], Sahab et al. [4], Singh [5–7] and many others.

2. Preliminaries

In the material to be presented here, the following definitions have been used:

In what follows, $(\mathcal{E}, \tau)$ will be a Hausdorff locally convex topological vector space. A family $\{p_\alpha : \alpha \in \Delta\}$ of seminorms defined on $\mathcal{E}$ is said to be an associated family of seminorms for τ if the family $\{\gamma\mathcal{U} : \gamma > 0\}$, where $\mathcal{U} = \bigcap_{i=1}^{n} \mathcal{U}_{\alpha_i}$, $n \in \mathbb{N}$, and $\mathcal{U}_{\alpha_i} = \{x \in \mathcal{E} : p_{\alpha_i}(x) \leq 1\}$, forms a base of neighbourhoods of zero for τ. A family $\{p_\alpha : \alpha \in \Delta\}$ of seminorms defined on $\mathcal{E}$ is called an augmented associated family for τ if $\{p_\alpha : \alpha \in \Delta\}$ is an associated family with the property that

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0893-9659/$ – see front matter © 2009 Published by Elsevier Ltd
doi:10.1016/j.aml.2009.06.025

the seminorm $\max\{p_\alpha, p_\beta\} \in \{p_\alpha : \alpha \in \Delta\}$ for any $\alpha, \beta \in \Delta$. The associated and augmented families of seminorms will be denoted by $\mathcal{A}(\tau)$ and $\mathcal{A}^*(\tau)$, respectively. It is well known that given a locally convex space $(\mathcal{E}, \tau)$, there always exists a family $\{p_\alpha : \alpha \in \Delta\}$ of seminorms defined of $\mathcal{E}$ such that $\{p_\alpha : \alpha \in \Delta\} = \mathcal{A}^*(\tau)$ (see [8, pp 203]). A subset $\mathcal{M}$ of $\mathcal{E}$ is τ-bounded if and only if each p_α is bounded on $\mathcal{M}$.

Suppose that $\mathcal{M}$ is a τ-bounded subset of $\mathcal{E}$. For this set $\mathcal{M}$, we can select a number $\lambda_\alpha > 0$ for each $\alpha \in \Delta$ such that $\mathcal{M} \subset \lambda_\alpha \mathcal{U}_\alpha$ where $\mathcal{U}_\alpha = \{x \in \mathcal{M} : p_\alpha(x) \leq 1\}$. Clearly, $\mathcal{B} = \bigcap_\alpha \lambda_\alpha \mathcal{U}_\alpha$ is τ-bounded, τ-closed, absolutely convex and contains $\mathcal{M}$. The linear span $\mathcal{E}_\mathcal{B}$ of $\mathcal{B}$ in $\mathcal{E}$ is $\bigcup_{n=1}^\infty n\mathcal{B}$. The Minkowski functional of $\mathcal{B}$ is a norm $\|\cdot\|_\mathcal{B}$ on $\mathcal{E}_\mathcal{B}$. Thus, $(\mathcal{E}_\mathcal{B}, \|\cdot\|_\mathcal{B})$ is a normed space with $\mathcal{B}$ as its closed unit ball and $\sup_\alpha p_\alpha(x/\lambda_\alpha) = \|x\|_\mathcal{B}$ for each $x \in \mathcal{E}_\mathcal{B}$. (for details, see [9,8,10]).

Definition 2.1 (*[9]*). Let $\mathcal{I}$ and $\mathcal{T}$ be self-maps on $\mathcal{M}$. The map $\mathcal{T}$ is called

(i) $\mathcal{A}^*(\tau)$-nonexpansive if for all $x, y \in \mathcal{M}$

$$p_\alpha(\mathcal{T}x - \mathcal{T}y) \leq p_\alpha(x - y),$$

for each $p_\alpha \in \mathcal{A}^*(\tau)$.

(ii) $\mathcal{A}^*(\tau)$-$\mathcal{I}$-nonexpansive if for all $x, y \in \mathcal{M}$

$$p_\alpha(\mathcal{T}x - \mathcal{T}y) \leq p_\alpha(\mathcal{I}x - \mathcal{I}y),$$

for each $p_\alpha \in \mathcal{A}^*(\tau)$.

For simplicity, we shall call $\mathcal{A}^*(\tau)$-nonexpansive ($\mathcal{A}^*(\tau) - \mathcal{I}$-nonexpansive) maps to be nonexpansive ($\mathcal{I}$-nonexpansive).

Definition 2.2 (*[11]*). A pair of self-mappings $(\mathcal{T}, \mathcal{I})$ of a locally convex space $(\mathcal{E}, \tau)$ is said to be compatible, if $p_\alpha(\mathcal{T}\mathcal{I}x_n - \mathcal{I}\mathcal{T}x_n) \to 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{E}$ such that $\mathcal{T}x_n, \mathcal{I}x_n \to t \in \mathcal{E}$.

Every commuting pair of mappings is compatible but the converse is not true in general.

Definition 2.3. Suppose that $\mathcal{M}$ is q-starshaped with $q \in \mathcal{F}(\mathcal{I})$ and is both $\mathcal{T}$- and $\mathcal{I}$-invariant. Then $\mathcal{T}$ and $\mathcal{I}$ are called $\mathcal{R}$-subcommuting [12–14] on $\mathcal{M}$, if for all $x \in \mathcal{M}$ and for all $p_\alpha \in \mathcal{A}^*(\tau)$, there exists a real number $\mathcal{R} > 0$ such that $p_\alpha(\mathcal{I}\mathcal{T}x - \mathcal{T}\mathcal{I}x) \leq (\frac{\mathcal{R}}{k})p_\alpha(((1-k)q + k\mathcal{T}x) - \mathcal{I}x)$ for each $k \in (0, 1)$. If $\mathcal{R} = 1$, then the maps are called 1-subcommuting. The $\mathcal{I}$ and $\mathcal{T}$ are called $\mathcal{R}$-subweakly commuting [15] on $\mathcal{M}$, if for all $x \in \mathcal{M}$ and for all $p_\alpha \in \mathcal{A}^*(\tau)$, there exists a real number $\mathcal{R} > 0$ such that $p_\alpha(\mathcal{I}\mathcal{T}x - \mathcal{T}\mathcal{I}x) \leq \mathcal{R}d_{p_\alpha}(\mathcal{I}x, [q, \mathcal{T}x])$, where $[q, x] = (1-k)q + kx : 0 \leq k \leq 1$.

Remark 2.4. (1) It is obvious that commutativity implies $\mathcal{R}$-subcommutativity, which in turn implies $\mathcal{R}$-weakly commutativity [13,14].

(2) It is also well known that commuting maps are $\mathcal{R}$-subweakly commuting maps and $\mathcal{R}$-subweakly commuting maps are $\mathcal{R}$-weakly commuting but not conversely in general (see [15]).

To clear the above remarks, in the following, we have furnished some examples:

Example 2.5. Let $\mathcal{X} = \mathbb{R}$ with norm $\|x\| = |x|$ and $\mathcal{M} = [1, \infty)$. Let $\mathcal{T}, \mathcal{S} : \mathcal{M} \to \mathcal{M}$ be defined by

$$\mathcal{T}x = x^2 \quad \text{and} \quad \mathcal{S}x = 2x - 1$$

for all $x \in \mathcal{M}$. Then $\mathcal{T}$ and $\mathcal{S}$ are $\mathcal{R}$-weakly commuting with $\mathcal{R} = 2$. However, they are not $\mathcal{R}$-subcommuting because

$$|\mathcal{T}\mathcal{S}x - \mathcal{S}\mathcal{T}x| \leq \left(\frac{\mathcal{R}}{k}\right)|(k\mathcal{T}x + (1-k)p) - \mathcal{S}x|$$

does not hold for $x = 2$ and $k = \frac{2}{3}$, where $p = 1 \in \mathcal{F}(\mathcal{S})$.

Example 2.6. Let $\mathcal{X} = \mathbb{R}$ with norm $\|x\| = |x|$ and $\mathcal{M} = [1, \infty)$. Let $\mathcal{T}, \mathcal{S} : \mathcal{M} \to \mathcal{M}$ be defined by

$$\mathcal{T}x = 4x - 3 \quad \text{and} \quad \mathcal{S}x = 2x^2 - 1$$

for all $x \in \mathcal{M}$. Then $\mathcal{M}$ is p-starshaped with $p = 1 \in \mathcal{F}(\mathcal{S})$ and is both $\mathcal{T}$ and $\mathcal{S}$-invariant. Also, $|\mathcal{T}\mathcal{S}x - \mathcal{S}\mathcal{T}x| = 24(x-1)^2$. Further,

$$|\mathcal{T}\mathcal{S}x - \mathcal{S}\mathcal{T}x| \leq \left(\frac{\mathcal{R}}{k}\right)|(k\mathcal{T}x + (1-k)p) - \mathcal{S}x|$$

for all $x \in \mathcal{M}$, where $\mathcal{R} = 12$ and $p = 1 \in \mathcal{F}(\mathcal{S})$. Thus, $\mathcal{T}$ and $\mathcal{S}$ are $\mathcal{R}$-subcommuting on $\mathcal{M}$ but are not commuting on $\mathcal{M}$.

Example 2.7. Let $\mathcal{X} = \mathbb{R}^2$ with norm $\|(x, y)\| = \max\{|x|, |y|\}$, and let $\mathcal{T}$ and $\mathcal{S}$ be defined by

$$\mathcal{T}(x, y) = (2x - 1, y^3) \quad \text{and} \quad \mathcal{S}(x, y) = (x^2, y^2)$$

for all $(x, y) \in \mathcal{X}$. Then $\mathcal{T}$ and $\mathcal{S}$ are $\mathcal{R}$-subweakly commuting on $\mathcal{M} = \{(x, y) : x \geq 1, y \geq 1\}$ but they are not commuting on $\mathcal{M}$.

Definition 2.8. Suppose that $\mathcal{M}$ is q-starshaped with $q \in \mathcal{F}(\mathcal{I})$. Define $\bigwedge_q(\mathcal{I}, \mathcal{T}) = \{\bigwedge(\mathcal{I}, \mathcal{T}_k) : 0 \le k \le 1\}$ where $\mathcal{T}_k x = (1-k)q + k\mathcal{T}x$ and $\bigwedge(\mathcal{I}, \mathcal{T}_k) = \{\{x_n\} \subset \mathcal{M} : lim_n \mathcal{I}x_n = lim_n \mathcal{T}_k x_n = t \in \mathcal{M} \Rightarrow lim_n p_\alpha(\mathcal{I}\mathcal{T}_k x_n - \mathcal{T}_k \mathcal{I}x_n) = 0\}$, for all sequences $\{x_n\} \in \bigwedge_q(\mathcal{I}, \mathcal{T})$. Then $\mathcal{I}$ and $\mathcal{T}$ are called subcompatible [16,17] if

$$\lim_n p_\alpha(\mathcal{I}\mathcal{T}x_n - \mathcal{T}\mathcal{I}x_n) = 0$$

for all sequences $x_n \in \bigwedge_q(\mathcal{I}, \mathcal{T})$.

Obviously, subcompatible maps are compatible but the converse does not hold, in general, as the following example shows.

Example 2.9. Let $\mathcal{X} = \mathbb{R}$ with usual norm and $\mathcal{M} = [1, \infty)$. Let $\mathcal{I}(x) = 2x - 1$ and $\mathcal{T}(x) = x^2$, for all $x \in \mathcal{M}$. Let $q = 1$. Then $\mathcal{M}$ is q-starshaped with $\mathcal{I}q = q$. Note that $\mathcal{I}$ and $\mathcal{T}$ are compatible. For any sequence $\{x_n\}$ in $\mathcal{M}$ with $lim_n x_n = 2$, we have, $lim_n \mathcal{I}x_n = lim_n \mathcal{T}_{\frac{2}{3}} x_n = 3 \in \mathcal{M} \Rightarrow lim_n \|\mathcal{I}\mathcal{T}_{\frac{2}{3}} x_n - \mathcal{T}_{\frac{2}{3}} \mathcal{I}x_n\| = 0$. However, $lim_n \|\mathcal{I}\mathcal{T}x_n - \mathcal{T}\mathcal{I}x_n\| \neq 0$. Thus $\mathcal{I}$ and $\mathcal{T}$ are not subcompatible maps.

Note that $\mathcal{R}$-subweakly commuting and $\mathcal{R}$-subcommuting maps are subcompatible. The following simple example reveals that the converse is not true, in general.

Example 2.10. Let $\mathcal{X} = \mathbb{R}$ with usual norm and $\mathcal{M} = [0, \infty)$. Let $\mathcal{I}(x) = \frac{x}{2}$ if $0 \le x < 1$ and $\mathcal{I}x = x$ if $x \ge 1$, and $\mathcal{T}(x) = \frac{1}{2}$ if $0 \le x < 1$ and $\mathcal{T}x = x^2$ if $x \ge 1$. Then $\mathcal{M}$ is 1-starshaped with $\mathcal{I}1 = 1$ and $\bigwedge_q(\mathcal{I}, \mathcal{T}) = \{\{x_n\} : 1 \le x_n < \infty\}$. Note that $\mathcal{I}$ and $\mathcal{T}$ are subcompatible but not $\mathcal{R}$-weakly commuting for all $\mathcal{R} > 0$. Thus $\mathcal{I}$ and $\mathcal{T}$ are neither $\mathcal{R}$-subweakly commuting nor $\mathcal{R}$-subcommuting maps.

Definition 2.11 (*[9]*). Let $x_0 \in \mathcal{E}$ and $\mathcal{M} \subseteq \mathcal{E}$. Then for $0 < a \le 1$, we define the set $\mathcal{D}_a$ of best $(\mathcal{M}, a)$-approximant to x_0 as follows:

$$\mathcal{D}_a = \{y \in \mathcal{M} : ap_\alpha(y - x_0) = d_{p_\alpha}(x_0, \mathcal{M}), \quad \text{for all } p_\alpha \in \mathcal{A}^*(\tau)\},$$

where

$$d_{p_\alpha}(x_0, \mathcal{M}) = \inf\{p_\alpha(x_0 - z) : z \in \mathcal{M}\}.$$

For $a = 1$, definition reduces to the set $\mathcal{D}$ of best $\mathcal{M}$-approximant to x_0.

Definition 2.12. The map $\mathcal{T} : \mathcal{M} \to \mathcal{E}$ is said to be demiclosed at 0 if for every net $\{x_n\}$ in $\mathcal{M}$ converging weakly to x and $\{\mathcal{T}x_n\}$ converging strongly to 0, we have $\mathcal{T}x = 0$.

Throughout, this paper $\mathcal{F}(\mathcal{T})$ (resp. $\mathcal{F}(\mathcal{I})$) denotes the fixed point set of mapping $\mathcal{T}$ (resp. $(\mathcal{I})$).

3. Main result

To prove the main result, a lemma is presented below:

Lemma 3.1. *Let $\mathcal{T}$ and $\mathcal{I}$ be compatible self-maps of a τ-bounded subset $\mathcal{M}$ of a Hausdorff locally convex space $(\mathcal{E}, \tau)$. Then $\mathcal{T}$ and $\mathcal{I}$ be compatible on $\mathcal{M}$ with respect to $\|\cdot\|_{\mathcal{B}}$.*

Proof. By hypothesis for each $p_\alpha \in \mathcal{A}^*(\tau)$,

$$p_\alpha(\mathcal{T}\mathcal{I}x_n - \mathcal{I}\mathcal{T}x_n) \to 0, \tag{3.1}$$

whenever $\{x_n\}$ is a sequence in $\mathcal{M}$ such that

$$p_\alpha(\mathcal{T}x_n - t) \to 0, \qquad p_\alpha(\mathcal{I}x_n - t) \to 0$$

for some $t \in \mathcal{M}$.

Taking supremum on both sides,

$$\sup_\alpha p_\alpha\left(\frac{\mathcal{T}\mathcal{I}x_n - \mathcal{I}\mathcal{T}x_n}{\lambda_\alpha}\right) \to 0$$

i.e.,

$$\|\mathcal{T}\mathcal{I}x_n - \mathcal{I}\mathcal{T}x_n\|_{\mathcal{B}} \to 0$$

whenever $\{x_n\}$ is a sequence in $\mathcal{M}$ such that

$$\sup_\alpha p_\alpha\left(\frac{\mathcal{T}x_n - t}{\lambda_\alpha}\right) \to 0, \qquad \sup_\alpha p_\alpha\left(\frac{\mathcal{I}x_n - t}{\lambda_\alpha}\right) \to 0,$$

i.e.,

$$\|\mathcal{T}x_n - t\|_{\mathcal{B}} \to 0, \qquad \|\mathcal{I}x_n - t\|_{\mathcal{B}} \to 0. \quad \Box$$

A technique of Tarafdar [10] to obtain the following common fixed point theorem which generalizes Theorem 1.1.

Theorem 3.2. *Let $\mathcal{M}$ be a nonempty τ-bounded, τ-sequentially complete and convex subset of a Hausdorff locally convex space $(\mathcal{E}, \tau)$. Let $\mathcal{T}$ and $\mathcal{I}$ be compatible self-maps of $\mathcal{M}$ such that $\mathcal{T}(\mathcal{X}) \subseteq \mathcal{I}(\mathcal{X})$, $\mathcal{I}$ is linear and nonexpansive, and satisfying*

$$p_\alpha(\mathcal{T}x - \mathcal{T}y) \leq ap_\alpha(\mathcal{I}x - \mathcal{I}y) + (1-a)\max\{p_\alpha(\mathcal{T}x - \mathcal{I}x), p_\alpha(\mathcal{T}y - \mathcal{I}y)\} \tag{3.2}$$

for all $x, y \in \mathcal{M}$ and $p_\alpha \in \mathcal{A}^*(\tau)$, and for some $a \in (0, 1)$, then $\mathcal{T}$ and $\mathcal{I}$ have a unique common fixed point.

Proof. Since the norm topology on $\mathcal{E}_{\mathcal{B}}$ has a base of neighbourhoods of zero consisting of τ-closed sets and $\mathcal{M}$ is τ-sequentially complete, therefore, $\mathcal{M}$ is a $\|\cdot\|_B$-sequentially complete subset of $(\mathcal{E}_{\mathcal{B}}, \|\cdot\|_{\mathcal{B}})$ (Theorem 1.2, [10]). By Lemma 3.1, $\mathcal{T}$ and $\mathcal{I}$ are $\|\cdot\|_{\mathcal{B}}$-compatible maps of $\mathcal{M}$. From (3.2), we obtain for $x, y \in \mathcal{M}$,

$$\sup_\alpha p_\alpha\left(\frac{\mathcal{T}x - \mathcal{T}y}{\lambda_\alpha}\right) \leq a \sup_\alpha p_\alpha\left(\frac{\mathcal{I}x - \mathcal{I}y}{\lambda_\alpha}\right) + (1-a)\max\left\{\sup_\alpha p_\alpha\left(\frac{\mathcal{T}x - \mathcal{I}x}{\lambda_\alpha}\right), \sup_\alpha p_\alpha\left(\frac{\mathcal{T}y - \mathcal{I}y}{\lambda_\alpha}\right)\right\}.$$

Thus

$$\|\mathcal{T}x - \mathcal{T}y\|_{\mathcal{B}} \leq a\|\mathcal{I}x - \mathcal{I}y\|_{\mathcal{B}} + (1-a)\max\{\|\mathcal{T}x - \mathcal{I}x\|_{\mathcal{B}}, \|\mathcal{T}y - \mathcal{I}y\|_{\mathcal{B}}\}. \tag{3.3}$$

Note that, if $\mathcal{I}$ is nonexpansive on a τ-bounded, τ-sequentially complete subset $\mathcal{M}$ of $\mathcal{E}$, then $\mathcal{I}$ is also nonexpansive with respect to $\|\cdot\|_{\mathcal{B}}$ and hence $\|\cdot\|_{\mathcal{B}}$-continuous [8]. A comparison of our hypothesis with that of Theorem 1.1 tells that we can apply Theorem 1.1 to $\mathcal{M}$ as a subset of $(\mathcal{E}_{\mathcal{B}}, \|\cdot\|_{\mathcal{B}})$ to conclude that there exists a unique $w \in \mathcal{M}$ such that $w = \mathcal{T}w = \mathcal{I}w$. $\Box$

Example 3.3. Let $\mathcal{X} = \mathbb{R}$ with usual norm and $\mathcal{M} = [0, 1]$. Let $\mathcal{T}(x) = 1$ for $0 \leq x \leq \frac{1}{2}$, and $\mathcal{T}(x) = 0$ for $\frac{1}{2} < x \leq 1$, $\mathcal{I}(x) = 0$ for $0 < x \leq \frac{1}{2}$, and $\mathcal{I}(x) = 1$ for $\frac{1}{2} < x \leq 1$. Then all the assumptions of Theorem 3.2 are satisfied, but $\mathcal{T}$ and $\mathcal{I}$ have no common fixed point.

Theorem 3.4. *Let $\mathcal{M}$ be a nonempty τ-bounded, τ-sequentially complete and convex subset of a Hausdorff locally convex space $(\mathcal{E}, \tau)$. Let $\mathcal{T}$ and $\mathcal{I}$ be self-maps of $\mathcal{M}$ such that $\mathcal{T}$ and $\mathcal{I}$ are subcompatible. Suppose that $\mathcal{T}$ and $\mathcal{I}$ satisfy (3.2), $\mathcal{I}$ is linear and nonexpansive, $\mathcal{I}(\mathcal{M}) = \mathcal{M}$, $q \in \mathcal{F}(\mathcal{I})$, then $\mathcal{T}$ and $\mathcal{I}$ have a common fixed point provided one of the following conditions holds:*

(i) *$\mathcal{M}$ is τ-sequentially compact and $\mathcal{T}$ is continuous;*
(ii) *$\mathcal{T}$ is a compact map;*
(iii) *$\mathcal{M}$ is weakly compact in $(\mathcal{E}, \tau)$, $\mathcal{I}$ is weakly continuous and $\mathcal{I} - \mathcal{T}$ is demiclosed at* 0.

Proof. Choose a monotonically nondecreasing sequence $\{k_n\}$ of real numbers such that $0 < k_n < 1$ and $\limsup k_n = 1$. For each $n \in \mathbb{N}$, define $\mathcal{T}_n : \mathcal{M} \to \mathcal{M}$ as follows:

$$\mathcal{T}_n x = k_n\mathcal{T}x + (1-k_n)q. \tag{3.4}$$

Obviously, for each n, $\mathcal{T}_n$ maps $\mathcal{M}$ into itself, since $\mathcal{M}$ is convex.

As $\mathcal{I}$ is linear, we can have

$$\mathcal{T}_m\mathcal{I}x_n = k_n\mathcal{T}\mathcal{I}x_n + (1-k_n)q$$

and

$$\mathcal{I}\mathcal{T}_m x = k_n\mathcal{I}\mathcal{T}x_n + (1-k_n)\mathcal{I}q.$$

The subcompatibility of $\mathcal{I}$ and $\mathcal{T}$ and $q \in \mathcal{F}(\mathcal{I})$ implies that

$$\begin{aligned} 0 &\leq \lim_n p_\alpha(\mathcal{T}_n\mathcal{I}x_m - \mathcal{I}\mathcal{T}_n x_m) \\ &\leq \lim_m k_n p_\alpha(\mathcal{T}\mathcal{I}x_m - \mathcal{I}\mathcal{T}x_m) + \lim_m (1-k_n)p_\alpha(q - \mathcal{I}q) \\ &= 0, \end{aligned}$$

for any $\{x_m\} \subset \mathcal{M}$ with $\lim_m \mathcal{T}_n x_m = \lim_m \mathcal{I}x_m = t \in \mathcal{M}$.

Hence $\{\mathcal{T}_n\}$ and $\mathcal{I}$ are compatible for each n and $x_n \in \mathcal{M}$ and $\mathcal{T}_n(\mathcal{M}) \subseteq \mathcal{M} = \mathcal{I}(\mathcal{M})$, $\mathcal{I}$ is linear and $q \in \mathcal{F}(\mathcal{I})$. Therefore $\mathcal{T}_n(\mathcal{M}) \subseteq \mathcal{I}(\mathcal{M})$.

For all $x, y \in \mathcal{M}$, $p_\alpha \in \mathcal{A}^*(\tau)$ and for all $j \geq n$, (n fixed), we obtain from (3.2) and (3.4) that

$$\begin{aligned}
p_\alpha(\mathcal{T}_n x - \mathcal{T}_n y) &= k_n p_\alpha(\mathcal{T}x - \mathcal{T}y) \leq k_j p_\alpha(\mathcal{T}x - \mathcal{T}y)\\
&\leq p_\alpha(\mathcal{T}x - \mathcal{T}y)\\
&\leq a p_\alpha(\mathcal{I}x - \mathcal{I}y) + (1-a)\max\{p_\alpha(Tx - Ix), p_\alpha(\mathcal{T}y - \mathcal{I}y)\}\\
&\leq a p_\alpha(\mathcal{I}x - \mathcal{I}y) + (1-a)\max\{p_\alpha(\mathcal{T}x - \mathcal{T}_n x) + p_\alpha(\mathcal{T}_n x - \mathcal{I}x), p_\alpha(\mathcal{T}y - \mathcal{T}_n y) + p_\alpha(\mathcal{T}_n y - \mathcal{I}y)\}\\
&\leq a p_\alpha(\mathcal{I}x - \mathcal{I}y) + (1-a)\max\{(1-k_n)p_\alpha(\mathcal{T}x - q)\\
&\quad + p_\alpha(\mathcal{T}_n x - \mathcal{I}x), (1-k_n)p_\alpha(\mathcal{T}y - q) + p_\alpha(\mathcal{T}_n y - \mathcal{I}y)\}.
\end{aligned}$$

Hence for all $j \geq n$, we have

$$\begin{aligned}
p_\alpha(\mathcal{T}_n x - \mathcal{T}_n y) &\leq a p_\alpha(\mathcal{I}x - \mathcal{I}y) + (1-a)\max\{(1-k_j)p_\alpha(\mathcal{T}x - q)\\
&\quad + p_\alpha(\mathcal{T}_n x - \mathcal{I}x), (1-k_j)p_\alpha(\mathcal{T}y - q) + p_\alpha(\mathcal{T}_n y - \mathcal{I}y)\}.
\end{aligned}\tag{3.5}$$

As $\lim k_j = 1$, from (3.5), for every $n \in \mathbb{N}$, we have

$$\begin{aligned}
p_\alpha(\mathcal{T}_n x - \mathcal{T}_n y) &= \lim_j p_\alpha(\mathcal{T}_n x - \mathcal{T}_n y)\\
&\leq \lim_j \{a p_\alpha(\mathcal{I}x - \mathcal{I}y) + (1-a)\max\{(1-k_j)p_\alpha(\mathcal{T}x - q)\\
&\quad + p_\alpha(\mathcal{T}_n x - \mathcal{I}x), (1-k_j)p_\alpha(\mathcal{T}y - q) + p_\alpha(\mathcal{T}_n y - \mathcal{I}y)\}\}.
\end{aligned}\tag{3.6}$$

This implies that for every $n \in \mathbb{N}$,

$$p_\alpha(\mathcal{T}_n x - \mathcal{T}_n y) \leq a p_\alpha(\mathcal{I}x - \mathcal{I}y) + (1-a)\max\{p_\alpha(\mathcal{T}_n x - \mathcal{I}x), p_\alpha(\mathcal{T}_n y - \mathcal{I}y)\},\tag{3.7}$$

for all $x, y \in \mathcal{M}$ and for all $p_\alpha \in \mathcal{A}^*(\tau)$.

Moreover, $\mathcal{I}$ being nonexpansive on $\mathcal{M}$, implies that $\mathcal{I}$ is $\|\cdot\|_{\mathcal{B}}$-nonexpansive and, hence, $\|\cdot\|_{\mathcal{B}}$-continuous. Since the norm topology on $\mathcal{E}_{\mathcal{B}}$ has a base of neighbourhoods of zero consisting of τ-closed sets and $\mathcal{M}$ is τ-sequentially complete, therefore, $\mathcal{M}$ is a $\|\cdot\|_{\mathcal{B}}$-sequentially complete subset of $(\mathcal{E}_{\mathcal{B}}, \|\cdot\|_{\mathcal{B}})$ (see proof in [10, Theorem 1.2]). Thus from Theorem 3.2, for every $n \in \mathbb{N}$, $\mathcal{T}_n$ and $\mathcal{I}$ have unique common fixed point x_n in $\mathcal{M}$, i.e.,

$$x_n = \mathcal{T}_n x_n = \mathcal{I}x_n,\tag{3.8}$$

for each $n \in \mathbb{N}$.

(i) As $\mathcal{M}$ is τ-sequentially compact and $\{x_n\}$ is a sequence in $\mathcal{M}$, so $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $x_m \to y \in \mathcal{M}$. As $\mathcal{I}$ and $\mathcal{T}$ are continuous and

$$x_m = \mathcal{I}x_m = \mathcal{T}_m x_m = k_m \mathcal{T}x_m + (1-k_m)q,$$

so it follows that $y = \mathcal{T}y = \mathcal{I}y$.

(ii) As $\mathcal{T}$ is compact and $\{x_n\}$ is bounded, so $\{\mathcal{T}x_n\}$ has a subsequence $\{\mathcal{T}x_m\}$ such that $\{\mathcal{T}x_m\} \to z \in \mathcal{M}$. Now we have

$$x_m = \mathcal{T}_m x_m = k_m \mathcal{T}x_m + (1-k_m)q.$$

Proceeding to the limit as $m \to \infty$ and using the continuity of $\mathcal{I}$ and $\mathcal{T}$, we have $\mathcal{I}z = z = \mathcal{T}z$.

(iii) The sequence $\{x_n\}$ has a subsequence $\{x_m\}$ converges to $u \in \mathcal{M}$. Since $\mathcal{I}$ is weakly continuous and so as in (i), we have $\mathcal{I}u = u$. Now,

$$x_m = \mathcal{I}x_m = \mathcal{T}_m x_m = k_m \mathcal{T}x_m + (1-k_m)q$$

implies that

$$\mathcal{I}x_m - \mathcal{T}x_m = (1-k_m)[q - \mathcal{T}x_m] \to 0$$

as $m \to \infty$. The demiclosedness of $\mathcal{I} - \mathcal{T}$ at 0 implies that $(\mathcal{I} - \mathcal{T})u = 0$. Hence $\mathcal{I}u = u = \mathcal{T}u$. This completes the proof. □

Example 3.5. Let $\mathcal{X} = \mathbb{R}^2$ and $\mathcal{M} = \{0, 1, 1 - \frac{1}{n-1} : n \in \mathbb{N}\}$ be endowed with usual metric. Define $\mathcal{T}1 = 0$ and $\mathcal{T}0 = \mathcal{T}(1 - \frac{1}{n-1}) = 1$ for all $n \in \mathbb{N}$. Clearly, $\mathcal{M}$ is not convex. Let $\mathcal{I}x = x$ for all $x \in \mathcal{M}$. Now $\mathcal{T}$ and $\mathcal{I}$ satisfy (3.2) together with all other conditions of Theorem 3.4(i) except the condition that $\mathcal{T}$ is continuous. Note that $\mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) = \emptyset$.

Example 3.6. Let $\mathcal{X} = \mathbb{R}^2$ be endowed with the norm defined by $\|(a, b)\| = |a| + |b|$, $(a, b) \in \mathbb{R}^2$.

(1) Let $\mathcal{M} = \mathcal{A} \cup \mathcal{B}$, where $\mathcal{A} = \{(a, b) \in \mathcal{X} : 0 \leq a \leq 1, 0 \leq b \leq 4\}$ and $\mathcal{B} = \{(a, b) \in \mathcal{X} : 2 \leq a \leq 3, 0 \leq b \leq 4\}$. Define $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ by

$$\mathcal{T}(a, b) = \begin{cases} (2, b) & \text{if } (a, b) \in \mathcal{A}\\ (1, b) & \text{if } (a, b) \in \mathcal{B}\end{cases}$$

and $\mathcal{I}(x) = x$ for all $x \in \mathcal{M}$. All the conditions of Theorem 3.4(ii) are satisfied except that $\mathcal{M}$ is not convex. Note that $\mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) = \emptyset$.

(2) $\mathcal{M} = \{(a, b) \in \mathcal{X} : 2 \le a < \infty, 0 \le b \le 1\}$ and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is defined by

$$\mathcal{T}(a, b) = \{(a + 1, b) : (a, b) \in \mathcal{M}\}.$$

Define $\mathcal{I}(x) = x$ for all $x \in \mathcal{M}$. All the conditions of Theorem 3.4(ii) are satisfied except that $\mathcal{T}(\mathcal{M})$ is compact. Note $\mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) = \emptyset$. Notice that $\mathcal{M}$, being convex and $\mathcal{T}$-invariant.

(3) If $\mathcal{M} = \{(a, b) \in \mathcal{X} : 0 \le a < 1, 0 \le b \le 1\}$ and $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is defined by

$$\mathcal{T}(a, b) = \left(\frac{a}{2}, \frac{b}{3}\right) \quad \text{and} \quad \mathcal{I}(x) = x \quad \text{for all } x \in \mathcal{M}.$$

All of the conditions of Theorem 3.4(ii) are satisfied except the fact that $\mathcal{M}$ is closed. However $\mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I}) = \emptyset$.

Example 3.7. Let $\mathcal{M} = \mathbb{R}^2$ be endowed with the norm defined by $\|(a, b)\| = |a| + |b|$, $(a, b) \in \mathbb{R}^2$. Define $\mathcal{T}$ and $\mathcal{I}$ on $\mathcal{M}$ as follows:

$$\mathcal{T}(x, y) = \left(\frac{1}{2}(x - 2), \frac{1}{2}(x^2 + y - 4)\right),$$

$$\mathcal{I}(x, y) = \left(\frac{1}{2}(x - 2), (x^2 + y - 4)\right).$$

Obviously, $\mathcal{T}$ is $\mathcal{I}$-nonexpansive but $\mathcal{I}$ is not linear. Moreover, $\mathcal{F}(\mathcal{T}) = \{-2, 0\}$, $\mathcal{F}(\mathcal{I}) = \{(-2, y) : y \in \mathbb{R}\}$ and the set of coincidence points of $\mathcal{I}$ and $\mathcal{T}$, that is $\mathcal{C}(\mathcal{I}, \mathcal{T}) = \{(x, y) : y = 4 - x^2, x \in \mathbb{R}\}$. Thus $(\mathcal{T}, \mathcal{I})$ is a continuous, which is not compatible pair, and $(-2, 0)$ is a common fixed point of $\mathcal{I}$ and $\mathcal{T}$.

An application of Theorem 3.4, we prove the following more general result in best approximation theory.

Theorem 3.8. *Let $\mathcal{T}$ and $\mathcal{I}$ be self-maps of a Hausdorff locally convex space $(\mathcal{E}, \tau)$ and $\mathcal{M}$ a subset of $\mathcal{E}$ such that $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$ and $x_0 \in \mathcal{F}(\mathcal{T}) \cap \mathcal{F}(\mathcal{I})$. Suppose that $\mathcal{I}$ is nonexpansive and linear on $\mathcal{D}_a$. Further, suppose $\mathcal{T}$ and $\mathcal{I}$ satisfy (3.2) for all $x, y \in \mathcal{D}'_a = \mathcal{D}_a \cup \{x_0\}$ and pair $(\mathcal{T}, \mathcal{I})$ are subcompatible on $\mathcal{D}_a$. If $\mathcal{D}_a$ is nonempty convex and $\mathcal{I}(\mathcal{D}_a) = \mathcal{D}_a$, then $\mathcal{T}$ and I have a common fixed point in $\mathcal{D}_a$ provided one of the following conditions holds:*

(i) *$\mathcal{D}_a$ is τ-sequentially compact;*
(ii) *$\mathcal{T}$ is a compact map;*
(iii) *$\mathcal{D}_a$ is weakly compact in $(\mathcal{E}, \tau)$, $\mathcal{I}$ is weakly continuous and $\mathcal{I} - \mathcal{T}$ is demiclosed at* 0.

Proof. First, we show that $\mathcal{T}$ is self-maps on $\mathcal{D}_a$, i.e., $\mathcal{T} : \mathcal{D}_a \to \mathcal{D}_a$. Let $y \in \mathcal{D}_a$, then $\mathcal{I}y \in \mathcal{D}_a$, since $\mathcal{I}(\mathcal{D}_a) = \mathcal{D}_a$. Also, if $y \in \partial\mathcal{M}$, then $\mathcal{T}y \in \mathcal{M}$, since $\mathcal{T}(\partial\mathcal{M}) \subseteq \mathcal{M}$. Now since $\mathcal{T}x_0 = x_0 = \mathcal{I}x_0$, so for each $p_\alpha \in \mathcal{A}^*(\tau)$, we have from (3.2)

$$\begin{aligned} p_\alpha(\mathcal{T}y - x_0) &= p_\alpha(\mathcal{T}y - \mathcal{T}x_0) \\ &\le ap_\alpha(\mathcal{I}y - \mathcal{I}x_0) + (1 - a)\max\{p_\alpha(\mathcal{T}y - \mathcal{I}y), p_\alpha(\mathcal{T}x_0 - \mathcal{I}x_0)\} \\ &\le ap_\alpha(\mathcal{I}y - x_0) + (1 - a)\max\{p_\alpha(\mathcal{T}y - x_0) + p_\alpha(\mathcal{I}y - x_0)\} \\ &= p_\alpha(\mathcal{I}y - x_0) + (1 - a)p_\alpha(\mathcal{T}y - x_0). \end{aligned}$$

So, we have

$$ap_\alpha(\mathcal{T}y - \mathcal{T}x_0) \le p_\alpha(\mathcal{I}y - x_0).$$

Now, $\mathcal{T}y \in \mathcal{M}$ and $\mathcal{I}y \in \mathcal{D}_a$, this implies that $\mathcal{T}y$ is also closest to x_0, so $\mathcal{T}y \in \mathcal{D}_a$. Consequently $\mathcal{T}$ and $\mathcal{I}$ are self-maps on $\mathcal{D}_a$. The conditions of Theorem 3.4((i)–(iii)) are satisfied and, hence, there exists a $\nu \in \mathcal{D}_a$ such that $\mathcal{T}\nu = \nu = \mathcal{I}\nu$. This completes the proof. □

Acknowledgments

The authors are thankful to the referees for their helpful corrections, comments and valuable suggestions in preparation of the paper.

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